

# TYPE-DECOMPOSITION OF A PSEUDO-EFFECT ALGEBRA

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**ABSTRACT.** The theory of direct decomposition of a centrally orthocomplete effect algebra into direct summands of various types utilizes the notion of a type-determining (TD) set. A pseudo-effect algebra (PEA) is a (possibly) non-commutative version of an effect algebra. In this article we develop the basic theory of centrally orthocomplete PEAs, generalize the notion of a TD set to PEAs, and show that TD sets induce decompositions of centrally orthocomplete PEAs into direct summands.

## 1. INTRODUCTION

The classic theorem stating that a von Neumann algebra decomposes uniquely as a direct sum of subalgebras of types I, II, and III, [22], [4, I, §8], has played a prominent role both in the development of the theory of von Neumann algebras and in the applications of this theory in mathematical physics. Analogous type-decomposition theorems were featured in subsequent work on various generalizations of von Neumann algebras, including studies of AW\*-algebras [18], Baer \*-rings [19], and JW-algebras [26]. For a von Neumann algebra  $A$ , and for the aforementioned generalizations thereof, the subset  $P$  of all projections (self-adjoint idempotents) in  $A$  forms an orthomodular lattice (OML) [1, 16], and the decomposition of  $A$  into types induces a corresponding direct decomposition of the OML  $P$ . Conversely, a direct decomposition of  $P$  yields a direct-sum decomposition of the enveloping algebra  $A$ . These connections between direct-sum decompositions of  $A$  and direct decompositions of  $P$  have motivated a number of studies of direct decompositions of more general OMLs.

The type-decomposition theorem for a von Neumann algebra is dependent on the von Neumann-Murray dimension theory; likewise, the early type-decomposition theorems for OMLs were corollaries of the lattice-based dimension theories of L. Loomis [20] and of S. Maeda [21]. The work of Loomis and Maeda was further developed by A. Ramsay [24] who proved that an arbitrary complete OML is uniquely decomposed into four special direct summands, one of which can be organized into a Loomis dimension lattice. More recent and considerably more general results on type-decomposition based on dimension theory can be found in the monograph of K. Goodearl and F. Wehrung [14].

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1991 *Mathematics Subject Classification.* Primary 06C15; Secondary 17C65, 46L45.

*Key words and phrases.* (pseudo-)effect algebra, (pseudo-)MV-algebra, orthomodular poset, orthomodular lattice, boolean algebra, von Neumann algebra, JW-algebra, Loomis dimension lattice, types I, II, and III..

The second and third authors were supported by Research and Development Support Agency under the contract No. APVV-0071-06 and LPP-0199-07; grant VEGA 2/0032/09, Center of Excellence SAS - Quantum Technologies; ERDF OP R&D Project CE QUTE ITMS 26240120009.

In [17, §7] G. Kalmbach, without employing lattice dimension theory *per se*, obtained decompositions of an arbitrary complete OML into direct summands with various special properties—moreover, Ramsay’s fourfold decomposition is a special case of Kalmbach’s theory. In [2], J. Carrega, G. Chevalier, and R. Mayet obtained the direct decompositions of Kalmbach and Ramsay by methods more in the spirit of universal algebra.

In [11], the decomposition theory of Kalmbach, Carrega, *et al.* was extended to the class of centrally orthocomplete effect algebras (COEAs) by employing the notion of a type-determining (TD) set. Effect algebras [9] are very general partially ordered algebraic structures, originally formulated as an algebraic base for the theory of measurement in quantum mechanics. Special cases of lattice-ordered effect algebras are OMLs and the MV-algebras of C. Chang [3].

The notion of a (possibly) non-commutative effect algebra, called a *pseudo-effect algebra*, was introduced and studied in a series of papers by A. Dvurečenskij and T. Vetterlein [6, 7, 5]. Whereas a prototypic example of an effect algebra is the order interval from 0 to a positive element in a partially ordered abelian group, the analogous interval in a partially ordered non-commutative group forms a pseudo-effect algebra.

We review the definition and some of the notation for a pseudo-effect algebra  $E$  in Section 2, and we study the center  $\Gamma(E)$  of  $E$  in Section 3. In Section 4, we focus attention on centrally orthocomplete pseudo-effect algebras (COPEAs) and define the *central cover* of an element in a COPEA. *For the remainder of the article, we assume that  $E$  is a COPEA.* The notion of a *type-determining* (TD) subset of  $E$  is introduced in Section 5, it is shown that TD subsets induce decompositions of  $E$  into direct summands of various types. The article ends with Section 6 where the important idea of a *type-class* of pseudo-effect algebras is introduced and a number of pertinent examples of type-classes and corresponding TD subsets of  $E$  are presented. Examples of the corresponding decompositions are given.

## 2. PSEUDO-EFFECT ALGEBRAS

A partial algebra  $(E; +, 0, 1)$ , where  $+$  is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* (PEA) iff, for all  $a, b, c \in E$ , the following conditions hold for all  $a, b, c \in E$ :

- (i)  $a + b$  and  $(a + b) + c$  exist iff  $b + c$  and  $a + (b + c)$  exist, and in this case  $(a + b) + c = a + (b + c)$ .
- (ii) There is exactly one  $d \in E$  and exactly one  $e \in E$  such that  $a + d = e + a = 1$ .
- (iii) If  $a + b$  exists, there are elements  $d, e \in E$  such that  $a + b = d + a = b + e$ .
- (iv) If  $1 + a$  or  $a + 1$  exists, then  $a = 0$ .

Suppose that  $E$  is a pseudo-effect algebra. If  $a, b \in E$ , define  $a \leq b$  iff there exists an element  $c \in E$  such that  $a + c = b$ ; then  $\leq$  is a partial ordering on  $E$  such that  $0 \leq a \leq 1$  for all  $a \in E$ . It is possible to show that  $a \leq b$  iff  $b = a + c = d + a$  for some  $c, d \in E$ . We write  $c =: a/b$  and  $d =: b \setminus a$ . Then  $(b \setminus a) + a = a + (a/b) = b$ , and  $a = (b \setminus a)/b = b \setminus (a/b)$ . If  $a \leq b \leq c$ , then

$$\begin{aligned} (c \setminus a) \setminus (b \setminus a) &= c \setminus b; \quad (a/b) / (a/c) = b/c; \\ (c \setminus b) / (c \setminus a) &= b \setminus a; \quad (a/c) \setminus (b/c) = a/b. \end{aligned}$$

We define  $x^- := 1 \setminus x$  and  $x^\sim := x / 1$  for any  $x \in E$ . For a given element  $e \in E$ , we denote the order interval from 0 to  $e$  by  $E[0, e] := \{x \in E : 0 \leq x \leq e\}$  and we define the partial binary operation  $+_e$  on  $E[0, e]$  as follows: for  $f, g \in E[0, e]$ ,  $f +_e g$  exists iff  $f + g$  exists in  $E$  and  $f + g \in E[0, e]$ , in which case  $f +_e g = f + g$ . Then  $(E[0, e]; +_e, 0, e)$  is a pseudo-effect algebra. For any  $x \in E[0, e]$  we have  $x^{-e} := e \setminus x$ ,  $x^{\sim e} := x / e$ , and  $e = x^{-e} + x = x + x^{\sim e}$ .

If  $a, b \in E$ , we write an existing least upper bound (respectively, greatest lower bound) of  $a$  and  $b$  in the partially ordered set  $E$  as  $a \vee b$  (respectively, as  $a \wedge b$ ). Similarly,  $\bigvee_{i \in I} e_i$  and  $\bigwedge_{i \in I} e_i$  denote, respectively, the least upper bound in  $E$  (if it exists) and the greatest lower bound in  $E$  (if it exists) of a family  $(e_i)_{i \in I} \subseteq E$ . Elements  $a, b \in E$  are *disjoint* iff  $a \wedge b = 0$ . We say that  $E$  is *lattice-ordered* iff  $a \vee b$  and  $a \wedge b$  exist in  $E$  for all  $a, b \in E$ .

**2.1. Example.** Let  $G$  be any partially ordered (not necessarily abelian) additively-written group, choose any element  $0 \leq u \in G$ , and let  $G[0, u] = \{g \in G : 0 \leq g \leq u\}$ . Then  $(G[0, u]; +, 0, u)$  is a pseudo-effect algebra if we restrict the group operation  $+$  to  $G[0, u]$ .

If  $x_1, x_2, \dots, x_n$  are elements of a pseudo-effect algebra  $E$ , we define the *orthosum*  $x_1 + x_2 + \dots + x_n$  by recurrence:  $x_1 + x_2 + \dots + x_n$  exists iff  $x_1 + x_2 + \dots + x_{n-1}$  and  $(x_1 + x_2 + \dots + x_{n-1}) + x_n$  exists, in which case we put  $x_1 + x_2 + \dots + x_n := (x_1 + x_2 + \dots + x_{n-1}) + x_n$ . Owing to associativity, we may omit parentheses, but the order of elements is important.

Let  $E$  and  $F$  be pseudo-effect algebras. A mapping  $\phi : E \rightarrow F$  is a *morphism* of pseudo-effect algebras (PEA-morphism) iff  $\phi(1_E) = 1_F$  (where  $1_E$  and  $1_F$  are the unit elements in  $E$  and  $F$ , respectively), and  $\phi(a) + \phi(b)$  exists whenever  $a + b$  exists, with  $\phi(a + b) = \phi(a) + \phi(b)$ . A morphism is an *isomorphism* of pseudo-effect algebras (PEA-isomorphism) iff it is a bijection and  $\phi^{-1}$  is also a morphism.

For more about basic properties of pseudo-effect algebras see [6, 7].

### 3. CENTRAL ELEMENTS OF PSEUDO-EFFECT ALGEBRAS

**3.1. Standing Assumption.** *In the sequel,  $(E; +, 0, 1)$  is a pseudo-effect algebra.*

**3.2. Definition.** [5, Definition 2.1] An element  $c$  of  $E$  is said to be *central* if there exists an isomorphism<sup>1</sup>

$$f_c : E \rightarrow E[0, c] \times E[0, c^\sim]$$

such that  $f_c(c) = (c, 0)$  and if  $f_c(x) = (x_1, x_2)$ , then  $x = x_1 + x_2$  for any  $x \in E$ .

We denote by  $\Gamma(E)$  the set of all central elements of  $E$ , and we refer to  $\Gamma(E)$  as the *center* of  $E$ . Clearly,  $0, 1 \in \Gamma(E)$ . In the next proposition, we collect some properties of central elements (see [5, Propositions 2.2, 2.4, and 2.5]).

**3.3. Proposition.** *Let  $c$  be a central element of  $E$ , and let  $f_c$  be the corresponding mapping from Definition 3.2. Then, for all  $x, y, x_1, x_2 \in E$ :*

- (i)  $f_c(c^\sim) = (0, c^\sim)$ .
- (ii) If  $x \leq c$ , then  $f_c(x) = (x, 0)$ .
- (iii)  $c \wedge c^\sim = 0$ .
- (iv) If  $y \leq c^\sim$  then  $f_c(y) = (0, y)$ .
- (v)  $c^\sim = c^-$ .

<sup>1</sup>With coordinatewise operations, the cartesian product of PEAs is again a PEA.

(vi)  $x \wedge c \in E$ ,  $x \wedge c^\sim \in E$ , and

$$f_c(x) = (x \wedge c, x \wedge c^\sim).$$

(vii) If  $f_c(x) = (x_1, x_2)$ , then  $x = x_1 \vee x_2$ ,  $x_1 \wedge x_2 = 0$ , and  $x_1 + x_2 = x$ .

(viii)  $x \wedge c = 0$  iff  $x \leq c^-$  iff  $x \leq c^\sim$  iff  $c \leq x^-$  iff  $c \leq x^\sim$ .

(ix)  $c + c \in E$  implies  $c = 0$ .

(x) Let  $c_1, c_2, \dots, c_n \in \Gamma(E)$ ,  $c_i \wedge c_j = 0$  for  $i \neq j$ , and  $c_1 + c_2 + \dots + c_n = 1$ . Then  $x = x \wedge c_1 + x \wedge c_2 + \dots + x \wedge c_n$ .

In view of Proposition 3.3 (v), if  $c \in \Gamma(E)$ , then we shall write  $c' := c^- = c^\sim$ . Also, we say that elements  $c, d \in \Gamma(E)$  are *orthogonal* iff  $c \wedge d = 0$ .

**3.4. Theorem.** [5, Theorem 2.3] If  $c, d \in \Gamma(E)$ , then  $c \wedge d$  exists in  $E$  and belongs to  $\Gamma(E)$ , and  $\Gamma(E) = (\Gamma(E); \wedge, \vee, ', 0, 1)$  is a Boolean algebra.

If  $c \in \Gamma(E)$ , then the mapping  $p_c : E \rightarrow E[0, c]$  defined by

$$p_c(x) := x \wedge c \text{ for all } x \in E$$

is a morphism from  $E$  onto  $E[0, c]$  whose kernel is  $E[0, c']$ .

**3.5. Proposition.** [5, Proposition 2.6] Let  $x \in E$  and  $c, d \in \Gamma(E)$ . Then:

- (i)  $p_{c \wedge d} = p_c p_d = p_d p_c$ .
- (ii) If  $c \wedge d = 0$ , then  $c + d = c \vee d = d + c$  and  $p_{c \vee d}(x) = p_c(x) + p_d(x) = p_d(x) + p_c(x)$ .
- (iii) If  $d \leq c$ , then  $c \setminus d = c \wedge d' = d/c$  and  $p_{c \wedge d'}(x) = p_c(x) \setminus p_d(x) = p_d(x)/p_c(x)$ .

**3.6. Theorem.** [5, Proposition 2.7] Let  $c_1, c_2, \dots, c_n \in \Gamma(E)$  with  $c_i \wedge c_j = 0$  for  $i \neq j$ . Then:

- (i)  $c := \bigvee_{i=1}^n c_i = c_1 + c_2 + \dots + c_n \in \Gamma(E)$ , and for all  $x \in E$ ,

$$x \wedge c = \bigvee_{i=1}^n (x \wedge c_i) = x \wedge c_1 + \dots + x \wedge c_n.$$

- (ii) If  $x_i \leq c_i$  for  $i = 1, 2, \dots, n$ , then  $x_1 + x_2 + \dots + x_n = x_1 \vee x_2 \vee \dots \vee x_n = x_{i_1} + x_{i_2} + \dots + x_{i_n}$ , where  $(i_1, i_2, \dots, i_n)$  is any permutation of  $(1, 2, \dots, n)$ .
- (iii) If  $a_1, a_2, \dots, a_n \in \Gamma(E)$ , then for all  $x \in E$ ,

$$x \wedge \left( \bigvee_{i=1}^n a_i \right) = \bigvee_{i=1}^n (x \wedge a_i).$$

**3.7. Theorem.** Suppose that  $c_1, c_2, \dots, c_n$  are pairwise orthogonal elements of  $\Gamma(E)$  with  $c_1 + c_2 + \dots + c_n = 1$ , let  $X := E[0, c_1] \times E[0, c_2] \times \dots \times E[0, c_n]$ , and define  $\Phi : X \rightarrow E$  by  $\Phi(e_1, e_2, \dots, e_n) := e_1 + e_2 + \dots + e_n = e_1 \vee e_2 \vee \dots \vee e_n$  for all  $(e_1, e_2, \dots, e_n) \in X$ . Then : (i)  $\Phi : X \rightarrow E$  is a PEA-isomorphism. (ii) If  $e \in E$ , then  $\Phi^{-1}(e) = (e \wedge c_1, e \wedge c_2, \dots, e \wedge c_n)$ .

*Proof.* If  $(e_1, e_2, \dots, e_n) \in X$ , then  $e_1 + e_2 + \dots + e_n = e_1 \vee e_2 \vee \dots \vee e_n$  by Theorem 3.6 (ii). Clearly,  $\Phi(1) = \Phi((c_1, c_2, \dots, c_n)) = c_1 + c_2 + \dots + c_n = 1$ . Assume that  $e := (e_1, e_2, \dots, e_n)$ ,  $f := (f_1, f_2, \dots, f_n) \in X$  are such that  $(e_1, e_2, \dots, e_n) + (f_1, f_2, \dots, f_n) = (e_1 + f_1, e_2 + f_2, \dots, e_n + f_n)$  exists in  $X$ . Then  $\Phi((e_1, e_2, \dots, e_n)) = e_1 + e_2 + \dots + e_n = e_1 \vee \dots \vee e_n$ ,  $\Phi((f_1, f_2, \dots, f_n)) = f_1 + f_2 + \dots + f_n = f_1 \vee \dots \vee f_n$ . Since  $e_i + f_i$  exists for  $i = 1, 2, \dots, n$ , we have  $e_i \leq f_i^-$  for  $i = 1, 2, \dots, n$ , and for

$i \neq j$ , we have  $e_i \leq c_i, f_j \leq c_j$ , whence  $e_i \leq c_i \leq c_j^- \leq f_j^-$ . Then  $\Phi(e) = \Phi((e_1, \dots, e_n)) = e_1 \vee e_2 \vee \dots \vee e_n \leq f_1^- \wedge f_2^- \wedge \dots \wedge f_n^- = \Phi(f)^-$ , so that  $\Phi(e) + \Phi(f)$  exists. Moreover, by associativity and Theorem 3.6 (ii),

$$\Phi((e_1, \dots, e_n)) + \Phi((f_1, \dots, f_n)) = (e_1 + e_2 + \dots + e_n) + (f_1 + f_2 + \dots + f_n) =$$

$$(e_1 + f_1) + (e_2 + f_2) + \dots + (e_n + f_n) = \Phi((e_1, e_2, \dots, e_n) + (f_1, f_2, \dots, f_n)).$$

This shows that  $\Phi$  is additive. For each  $e \in E$ , define  $\Psi : E \rightarrow X$  by  $\Psi(e) := (e \wedge c_1, e \wedge c_2, \dots, e \wedge c_n) = (p_{c_1}(e), \dots, p_{c_n}(e))$ . Clearly,  $\Psi(1) = 1$  in  $X$ , and if  $e + f$  exists, then  $\Psi(e + f) = \Psi(e) + \Psi(f)$ , since  $p_{c_i}$  are morphisms for all  $i$ . Then  $\Phi \circ \Psi(e) = e \wedge c_1 + e \wedge c_2 + \dots + e \wedge c_n = e$  by Proposition 3.3 (x). If  $(e_i)_{i=1}^n \subseteq X$ , then  $\Psi \circ \Phi((e_i)_{i=1}^n) = \Psi((e_1 + e_2 + \dots + e_n)) = (p_{c_i}(e_1 + \dots + e_n))_{i=1}^n = (e_i)_{i=1}^n$ , since  $p_{c_i}, i = 1, 2, \dots, n$  is a morphism, and  $e_i \leq c_j$  for  $i = j$ , while  $e_i \leq c_j'$  if  $i \neq j$ . It follows that  $\Phi^{-1} = \Psi$ , and  $\Psi$  is a morphism, hence  $\Phi$  is an isomorphism.  $\square$

**3.8. Theorem.** [5, Proposition 2.8] *For all  $c \in \Gamma(E)$ ,  $\Gamma(E[0, c]) = \Gamma(E)[0, c]$ .*

**3.9. Lemma.** *Suppose that  $e \in E$ ,  $(f_i)_{i \in I} \subseteq E$ ,  $e + f_i$  (respectively,  $f_i + e$ ) exists for all  $i \in I$ , and  $\bigvee_{i \in I} f_i$  exists in  $E$ . Then  $\bigvee_{i \in I} (e + f_i)$  (respectively,  $\bigvee_{i \in I} (f_i + e)$ ) exists in  $E$  and  $e + \bigvee_{i \in I} f_i = \bigvee_{i \in I} (e + f_i)$  (respectively,  $\bigvee_{i \in I} f_i + e = \bigvee_{i \in I} (f_i + e)$ ).*

*Proof.* Let  $f := \bigvee_{i \in I} f_i$ . Assume that  $e + f_i$  exists for all  $i \in I$ . Then  $f_i \leq e^\sim$  for all  $i \in I$ , so that  $f \leq e^\sim$ . Also  $e + f_i \leq e + f$  for all  $i \in I$ . Suppose that  $r \in E$  and  $e + f_i \leq r$  for all  $i \in I$ . It suffices to prove that  $e + f \leq r$ . We have  $e \leq e + f_i \leq r = e + (e/r)$ , whence  $f_i \leq e/r$  for all  $i \in I$ , and it follows that  $f \leq e/r$ , hence  $e + f \leq r$ . Now assume that  $f_i + e$  exists for all  $i \in I$ . Then  $f_i \leq e^-$ , whence  $f \leq e^-$ . Then  $f_i + e \leq f + e$ , and let  $r \in E$  be such that  $f_i + e \leq r$  for all  $i \in I$ . Then  $f_i \leq r \setminus e$  for all  $i \in I$ , whence  $f \leq r \setminus e$ , and this implies  $f + e \leq r$ .  $\square$

**3.10. Lemma.** *Suppose that  $\phi : E \rightarrow E$  satisfies the conditions  $\phi(e) + f$  exists  $\Rightarrow e + \phi(f)$  exists, and  $f + \phi(e)$  exists  $\Rightarrow \phi(f) + e$  exists for all  $e, f \in E$ . Then (i)  $\phi$  is order preserving. (ii) If  $(e_i)_{i \in I} \subseteq E$  and  $e := \bigvee e_i$  exists in  $E$ , then  $\bigvee \phi(e_i)$  exists in  $E$  and  $\phi(e) = \bigvee_{i \in I} \phi(e_i)$ .*

*Proof.* (i) Suppose  $e \leq f$ . Then  $f^\sim \leq e^\sim$ , and as  $\phi(f) + \phi(f)^\sim$  exists, then  $f + \phi(\phi(f)^\sim)$  exists  $\Rightarrow \phi(\phi(f)^\sim) \leq f^\sim \leq e^\sim \Rightarrow e + \phi(\phi(f)^\sim)$  exists  $\Rightarrow \phi(e) + \phi(f)^\sim$  exists  $\Rightarrow \phi(e) \leq \phi(f)$ . (ii) Assume the hypothesis of (ii). As  $e_i \leq e$ , it follows from (i) that  $\phi(e_i) \leq \phi(e)$  for all  $i \in I$ . Suppose that  $f \in E$  and  $\phi(e_i) \leq f$  for all  $i \in I$ . Then  $\phi(e_i) + f^\sim$  exists  $\Rightarrow e_i + \phi(f^\sim)$  exists  $\Rightarrow e_i \leq (\phi(f^\sim))^- \Rightarrow e \leq (\phi(f^\sim))^- \Rightarrow e + \phi(f^\sim)$  exists  $\Rightarrow \phi(e) + f^\sim$  exists  $\Rightarrow \phi(e) \leq f$ , proving (ii).  $\square$

**3.11. Theorem.** *Let  $c \in \Gamma(E)$  and let  $(e_i)_{i \in I}$  be a family of elements of  $E$ . Then:*

- (i) *If  $\bigvee_{i \in I} e_i$  exists in  $E$ , then  $c \wedge \bigvee_{i \in I} e_i = \bigvee_{i \in I} (c \wedge e_i)$ .*
- (ii) *For every  $e \in E$ ,  $c = c \wedge e + c \wedge e^\sim$ .*

*Proof.* (i) Define  $\phi : E \rightarrow E$  by  $\phi(e) := c \wedge e$  for all  $e \in E$ . Suppose  $e, f \in E$  and assume that  $\phi(e) + f$  exists. Then  $c \wedge e \leq f^- \leq (c \wedge f)^-$ . Also,  $c^- \wedge e \leq c^- \vee f^- = (c \wedge f)^-$ , and by Proposition 3.3 (vi) and (vii),  $e = (c \wedge e) \vee (c' \wedge e) \leq (c \wedge f)^-$ , and consequently,  $e + \phi(f)$  exists. Now assume that  $f + \phi(e)$  exists, then  $c \wedge e \leq f^\sim \leq (c \wedge f)^\sim$ , and  $c^\sim \wedge e \leq c^\sim \vee f^\sim = (c \wedge f)^\sim$ , and consequently  $e = (c \wedge e) \vee (c' \wedge e) \leq (c \wedge f)^\sim$ , and so  $\phi(f) + e$  exists. Therefore (i) follows from Lemma 3.10.

(ii) Put  $e_1 = e, e_2 = e^\sim$ . Then  $e_1 + e_2 = 1$ , and  $c = p_c(e_1 + e_2) = p_c(e_1) + p_c(e_2) = c \wedge e + c \wedge e^\sim$ .  $\square$

In the next theorem, we give an intrinsic characterization of central elements. (For a similar result see [27]).

**3.12. Theorem.** *An element  $c$  in a PEA  $E$  is central if and only if the following properties are satisfied:*

- (i) *For all  $a \in E$ , there are  $a_1, a_2 \in E$ ,  $a_1 \leq c$ ,  $a_2 \leq c^\sim$  and  $a = a_1 + a_2$ .*
- (ii) *If  $a, b \leq c$  (respectively,  $a, b \leq c^\sim$ ) and  $a + b$  is defined, then  $a + b \leq c$  (respectively,  $a + b \leq c^\sim$ );*
- (iii) *If  $x, y \in E$ ,  $x \leq c$ ,  $y \leq c^\sim$ , then  $x + y = y + x$ .*

*Proof.* Observe first that (i)–(iii) imply that  $c^\sim = c^-$  and  $c \wedge c^\sim = 0$ . Indeed, by (iii),  $1 = c + c^\sim = c^\sim + c$ , whence  $c^\sim = c^-$ . If  $x \leq c$ ,  $x \leq c^\sim$ , then by (ii),  $c + x \leq c$ , whence  $x = 0$ . If  $c$  is central then property (i) follows by the definition of central elements.

(ii): Let  $a, b \leq c$ , and  $a + b$  exist. Then  $f_c(a) = (a, 0)$ ,  $f_c(b) = (b, 0)$  and  $f_c(a + b) = (a, 0) + (b, 0) = (a + b, 0)$ , hence  $a + b \leq c$ . Part (iii) follows by Theorem 3.6 (ii).

To prove the converse, define  $f_c : E \rightarrow E[0, c] \times E[0, c^\sim]$  by  $f_c(a) = (a_1, a_2)$  when  $a = a_1 + a_2$ ,  $a_1 \leq c$ ,  $a_2 \leq c^\sim$  by (i). We shall prove that  $f_c$  satisfies Definition 3.2 in the following steps.

(1) Assume that for  $a \in E$ ,  $a = a_1 + a_2 = b_1 + b_2$  with  $a_1, b_1 \leq c$ ,  $a_2, b_2 \leq c^\sim$  be two decompositions of  $a$  by (i), and let  $a^\sim = d_1 + d_2$ ,  $d_1 \leq c$ ,  $d_2 \leq c^\sim$  be any decomposition of  $a^\sim$ . Then  $1 = a + a^\sim = (a_1 + a_2) + (d_1 + d_2) = a_1 + (a_2 + d_1) + d_2$  by associativity. Since  $a_2 \leq c^\sim$  and  $d_1 \leq c$ , we obtain by (iii) that  $a_2 + d_1 = d_1 + a_2$ . Again by associativity,  $1 = (a_1 + d_1) + (a_2 + d_2) = c + c^\sim$ , where  $a_1 + d_1 \leq c$ ,  $a_2 + d_2 \leq c^\sim$  by (ii). It follows that  $a_1 + d_1 = c$ ,  $a_2 + d_2 = c^\sim$ , so  $a_1 = c \setminus d_1$ ,  $a_2 = c^\sim \setminus d_2$ . Repeating this reasoning with  $a_1, a_2$  replaced by  $b_1, b_2$ , we obtain  $a_1 = b_1$ ,  $a_2 = b_2$ . This proves that  $f_c$  is well defined.

Clearly,  $f_c(c) = (c, 0)$  and if  $x \in E$  with  $f_c(x) = (x_1, x_2)$ , then  $x = x_1 + x_2$ . If  $f_c(a) = f_c(b)$ , then  $(a_1, a_2) = (b_1, b_2)$ , which implies  $a_1 = b_1$ ,  $a_2 = b_2$ , whence  $a = b$ . This shows that  $f_c$  is injective.

(2) Let  $a, b \in E$  be such that  $a + b$  exists. Let  $f_c(a) = (a_1, a_2)$ ,  $f_c(b) = (b_1, b_2)$ , and  $f_c(a + b) = (d_1, d_2)$ . Then  $a = a_1 + a_2$ ,  $b = b_1 + b_2$ ,  $a + b = d_1 + d_2$ . It follows that  $(a_1 + a_2) + (b_1 + b_2) = a_1 + b_1 + a_2 + b_2 = d_1 + d_2$ , using (iii). By (1) then,  $d_1 = a_1 + b_1$ ,  $d_2 = a_2 + b_2$ . Therefore  $f_c(a + b) = (d_1, d_2) = (a_1 + b_1, a_2 + b_2) = (a_1, a_2) + (b_1, b_2) = f_c(a) + f_c(b)$ . This proves that  $f_c$  is additive.

(3) Assume that  $f_c(a) + f_c(b)$  exists in  $E[0, c] \times E[0, c^\sim]$ . Then  $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ , whence  $a_1 + b_1$ ,  $a_2 + b_2$  exist in  $E$ , and hence  $(a_1 + b_1) + (a_2 + b_2) = a_1 + a_2 + b_1 + b_2 = a + b$ . It follows that  $a + b$  exists iff  $f_c(a) + f_c(b)$  exists. To prove surjectivity, let  $(x, y) \in E[0, c] \times E[0, c^\sim]$ . Put  $z = x + y$ , then  $f_c(z) = (x, y)$ .

Paerts (1), (2) and (3) imply that  $f_c$  is a bijection such that  $f_c(a) + f_c(b)$  exists iff  $a + b$  exists, and  $f_c(a + b) = f_c(a) + f_c(b)$ , hence it is an isomorphism.  $\square$

**3.13. Lemma.** *If  $p \in E$ , then: (i)  $c \in \Gamma(E) \Rightarrow p \wedge c \in \Gamma(E[0, p])$ . (ii) The mapping  $c \mapsto p \wedge c$  for  $c \in \Gamma(E)$  is a boolean homomorphism of  $\Gamma(E)$  into the center  $\Gamma(E[0, p])$  of  $E[0, p]$ .*

*Proof.* (i) Let  $e \in E[0, p]$ . Then  $e = e \wedge c + e \wedge c' = e \wedge p \wedge c + e \wedge p \wedge c'$ . From  $p = p \wedge c + p \wedge c' = p \wedge c' + p \wedge c$ , we find that  $p \wedge c' = (p \wedge c) / p = p \setminus (p \wedge c)$ .

This implies that  $(p \wedge c)^{\sim p} = (p \wedge c)^{-p} = p \wedge c'$ . Moreover, for every  $e \in E[0, p]$  we have a decomposition  $e = e_1 + e_2$ ,  $e_1 \leq p \wedge c$ ,  $e_2 \leq p \wedge c' = (p \wedge c)/p$ . If  $e, f \leq p \wedge c$ , and  $e + f$  exists in  $E[0, p]$ , then  $e + f \leq p$ , and  $e, f \leq c$  implies  $e + f \leq c$ , hence  $e + f \leq p \wedge c$ . The same argument holds if  $e, f \leq (p \wedge c)^{\sim p} := p \wedge c'$ . If  $x \leq p \wedge c$ ,  $y \leq p \wedge c'$ , then  $x \leq c$ ,  $y \leq c'$  implies that  $x + y = y + x$ . This proves that  $p \wedge c \in \Gamma(E[0, p])$ . (ii) Part (ii) follows from Proposition 3.5.  $\square$

#### 4. CENTRALLY ORTHOCOMPLETE PEAS

**4.1. Definition.** Two elements  $p, q \in E$  are said to be  $\Gamma$ -orthogonal iff there are orthogonal central elements  $c, d \in \Gamma(E)$  such that  $p \leq c$  and  $q \leq d$ . A family  $(e_i)_{i \in I}$  is  $\Gamma$ -orthogonal iff there is a pairwise orthogonal family of elements  $(c_i)_{i \in I} \subseteq \Gamma(E)$  of central elements in  $E$  such that  $e_i \leq c_i$  for all  $i \in I$ .

Observe that, owing to Theorem 3.6 (ii), if  $e_1, e_2, \dots, e_n$  are pairwise  $\Gamma$ -orthogonal elements, then their orthosum exists and does not depend on the order of its summands; moreover,  $\sum_{i=1}^n e_i = e_1 + e_2 + \dots + e_n = e_1 \vee e_2 \vee \dots \vee e_n$ .

**4.2. Definition.** Let  $(e_i)_{i \in I}$  be a  $\Gamma$ -orthogonal family in  $E$  and let  $\mathcal{F}$  be the collection of all finite subsets of the indexing set  $I$ . Then  $(e_i)_{i \in I}$  is *orthosummable* iff

$$\sum_{i \in I} e_i := \bigvee_{F \in \mathcal{F}} \sum_{i \in F} e_i$$

exists in  $E$ , in which case we refer to  $\sum_{i \in I} e_i$  as the *orthosum* of the family. By definition,  $E$  is a *centrally orthocomplete pseudo-effect algebra* (COPEA) iff every  $\Gamma$ -orthogonal family in  $E$  is orthosummable.

**4.3. Lemma.** (i) If  $e$  and  $f$  are  $\Gamma$ -orthogonal elements of  $E$ , then  $e \leq f \Rightarrow e = 0$ . (ii) A family of central elements is  $\Gamma$ -orthogonal iff it is pairwise orthogonal iff it is pairwise disjoint. (iii) Every finite  $\Gamma$ -orthogonal family in  $E$  is orthosummable and its orthosum is its supremum in  $E$ . (iv) An arbitrary  $\Gamma$ -orthogonal family in  $E$  is orthosummable iff it has an orthosum iff it has a supremum in  $E$ , and if it is orthosummable, then its orthosum coincides with its supremum. (v)  $E$  is a COPEA iff every  $\Gamma$ -orthogonal family in  $E$  has a supremum in  $E$ .

*Proof.* (i) If  $e, f \in E$  and  $c, d \in \Gamma(E)$  with  $e \leq c$  and  $f \leq d \leq c'$ , then  $e \leq f \Rightarrow e \leq c \wedge c' = 0$  by Proposition 3.3 (iii) and (v). (ii) Follows directly from the definitions of  $\Gamma$ -orthogonality and orthogonality of central elements. (iii) Follows from Theorem 3.6 (ii). (iv) Follows by (iii) and the definition of the orthosum. (v) Follows from (iv).  $\square$

**4.4. Standing Assumption.** In the sequel, we assume that  $E$  is a COPEA.

**4.5. Theorem.** Let  $(c_i)_{i \in I}$  be a pairwise orthogonal family of elements in  $\Gamma(E)$ , and let  $(e_i)_{i \in I}, (f_i)_{i \in I}$  be families in  $E$  such that  $e_i, f_i \leq c_i$  and  $e_i + f_i$  exists for all  $i \in I$ . Then: (i)  $c := \sum_{i \in I} c_i = \bigvee_{i \in I} c_i$ ,  $e := \sum_{i \in I} e_i = \bigvee_{i \in I} e_i \leq c$ ,  $f := \sum_{i \in I} f_i = \bigvee_{i \in I} f_i \leq c$ , and  $e + f$  exists. (ii)  $e + f = \sum_{i \in I} (e_i + f_i) = \bigvee_{i \in I} (e_i + f_i) \leq c$ .

*Proof.* (i) Part (i) follows from parts (ii) and (iv) of Lemma 4.3. For instance, the existence of  $e + f$  is proved as follows. As  $e_i + f_i$  exists for all  $i \in I$ , we have  $e_i \leq f_i^-$ . If  $i \neq j$ , then  $e_i \leq c_i, f_j \leq c_j, c_i \wedge c_j = 0$ , hence  $e_i + f_j$  exists, so that  $e_i \leq f_j^-$ . Then  $e = \bigvee_{i \in I} e_i \leq f_j^- \forall j \in I$ , whence  $e \leq \bigwedge_{j \in I} f_j^- = (\bigvee_{j \in I} f_j)^- = f^-$ , hence  $e + f$  exists.

(ii) If  $i \in I$ , then  $e_i, f_i \leq c_i$  implies that  $e_i + f_i \leq c_i$  by Theorem 3.12 (ii). From this it follows that  $(e_i + f_i)_{i \in I}$  is a  $\Gamma$ -orthogonal family in  $E$ , so by Lemma 4.3 (iv) and (v),

$$\sum_{i \in I} (e_i + f_i) = \bigvee_{i \in I} (e_i + f_i) \leq \bigvee_{i \in I} c_i = c.$$

By Lemma 3.9,  $e + f = (\bigvee_{s \in I} e_s) + f = \bigvee_{s \in I} (e_s + f)$ , and for each  $s \in I$ ,  $e_s + f = e_s + \bigvee_{t \in I} f_t = \bigvee_{t \in I} (e_s + f_t)$ , and so

$$\bigvee_{i \in I} (e_i + f_i) \leq \bigvee_{s, t \in I} (e_s + f_t) = e + f.$$

Suppose  $s, t \in I$ . If  $s = t$ , then  $e_s + f_t = e_s + f_s \leq \bigvee_{i \in I} (e_i + f_i)$ . If  $s \neq t$ , then  $e_s + f_t \leq (e_s + f_s) + (e_t + f_t) = (e_s + f_s) \vee (e_t + f_t)$ . Consequently,

$$e + f = \bigvee_{s, t \in I} (e_s + f_t) \leq \bigvee_{i \in I} (e_i + f_i).$$

Combining the results obtained above, we get (ii).  $\square$

**4.6. Corollary.** *Let  $(c_i)_{i \in I}$  be a pairwise orthogonal family of elements in  $\Gamma(E)$  and let  $d \in E$ . Put  $c := \bigvee_{i \in I} c_i$ ,  $e := \bigvee_{i \in I} (d \wedge c_i)$ , and  $f := \bigvee_{i \in I} (d^\sim \wedge c_i)$ . Then: (i)  $e \leq d$ ,  $f \leq d^\sim$ , and  $c = e + f$ . (ii) If  $d \in E[0, c]$ , then  $d = \sum_{i \in I} (d \wedge c_i) = \bigvee_{i \in I} (d \wedge c_i)$ .*

*Proof.* In Theorem 4.5, let  $e_i := d \wedge c_i$  and  $f_i := d^\sim \wedge c_i$ . (i) As  $e_i \leq d$  and  $f_i \leq d^\sim$  for all  $i \in I$ , we get  $e = \bigvee_{i \in I} e_i \leq d$ , and  $f = \bigvee_{i \in I} f_i \leq d^\sim$ . By Theorem 3.11 (ii),  $e_i + f_i = c_i$  for all  $i \in I$ , whence by Theorem 4.5 (ii),  $e + f = \bigvee_{i \in I} (e_i + f_i) = \bigvee_{i \in I} c_i = c$ . (ii) Assume that  $d \in E[0, c]$ . Then  $e \leq d \leq c$  by (i). Thus  $e \leq (d^\sim)^\sim$ , hence  $e + d^\sim$  exists, and  $e + d^\sim = \bigvee_{i \in I} (e_i + d^\sim)$  by Lemma 3.9. As  $c_i \in \Gamma(E)$ , we have

$$e_i + d^\sim = (d \wedge c_i) + d^\sim = (d \wedge c_i) + (d^\sim \wedge c_i) + (d^\sim \wedge c_i^\sim) = c_i + (d^\sim \wedge c_i^\sim)$$

$$= c_i \vee (d^\sim \wedge c_i^\sim) = c_i \vee (d^\sim \wedge c_i) \vee (d^\sim \wedge c_i^\sim) = c_i \vee d^\sim, \text{ so}$$

$$e + d^\sim = \bigvee_{i \in I} (d^\sim \vee c_i) \geq \bigvee_{i \in I} (c_i^\sim \vee c_i) = c^\sim \vee c = 1 = e + e^\sim.$$

By cancellation,  $d^\sim \geq e^\sim$ , whence  $d \leq e$ , and we have  $e = d$ .  $\square$

**4.7. Theorem.** (1) *Let  $(c_i)_{i \in I}$  be a pairwise orthogonal family of central elements, let  $c := \bigvee_{i \in I} c_i$ . Then  $c \in \Gamma(E)$ , and  $\Gamma(E)$  is a complete boolean algebra.* (2) *For each  $e \in E$  there is a smallest element  $d \in \Gamma(E)$  such that  $e \leq d$ .*

*Proof.* (1) We have to prove properties (i)–(iii) of Theorem 3.12 for  $c$ .

(i) Let  $d \in E$ . By Corollary 4.6,  $c = e + f$ ,  $e \leq d$ ,  $f \leq d^\sim$ . Then  $d = e + e/d$ , and  $e/d = \bigvee_{i \in I} (d \wedge c_i)/d \leq d \wedge c_i/d$  for all  $i \in I$ . Let  $x \in E$  be such that  $x \leq d \wedge c_i/d$  for all  $i \in I$ . Then  $d \wedge c_i + x \leq d$ , hence  $d \wedge c_i \leq d \setminus x$ , so  $\bigvee_{i \in I} (d \wedge c_i) \leq d \setminus x$ , and therefore  $x \leq \bigvee_{i \in I} (d \wedge c_i)/d$ . This proves that  $\bigvee_{i \in I} (d \wedge c_i)/d = \bigwedge_{i \in I} (d \wedge c_i/d) = \bigwedge_{i \in I} d \wedge c_i^\sim \leq \bigwedge_{i \in I} c_i^\sim = (\bigvee_{i \in I} c_i)^\sim = c^\sim$ . Finally we obtain  $d = e + e/d$ ,  $e \leq c$ ,  $e/d \leq c^\sim$ . (ii) Let  $e, f \leq c$  and suppose  $e + f$  exists. Then  $e_i := e \wedge c_i \leq c_i$ ,  $f_i := f \wedge c_i \leq c_i$ ,  $(e_i)_{i \in I}, (f_i)_{i \in I}$  are  $\Gamma$ -orthogonal, and  $e_i + f_i$  exists for all  $i \in I$ . By Theorem 4.5,  $e = \bigvee_{i \in I} e_i$ ,  $f = \bigvee_{i \in I} f_i$ , and  $e + f = \bigvee_{i \in I} (e_i + f_i) \leq c$ . Let  $e, f \leq c^\sim$  and suppose  $e + f$  exists. From  $c^\sim = (\bigvee_{i \in I} c_i)^\sim = \bigwedge_{i \in I} c_i^\sim$  we obtain that  $e, f \leq c_i^\sim$  for all  $i \in I$ , and since  $c_i$



is central,  $e + f \leq c_i^\sim$  for all  $i \in I$ . It follows that  $e + f \leq \bigwedge_{i \in I} c_i^\sim = c^\sim$ . (iii) Let  $x, y \in E$ ,  $x \leq c$ ,  $y \leq c^\sim$ . Then  $x \wedge c_i \leq c_i$ ,  $y \leq c^\sim \leq c_i^\sim$  for all  $i \in I$ , and  $x = \bigvee_{i \in I} x \wedge c_i$  by Theorem 3.11. Since  $c_i$  is central, we have  $x \wedge c_i + y = y + x \wedge c_i$ , and by Lemma 3.9,  $x + y = \bigvee_{i \in I} (x \wedge c_i + y) = \bigvee_{i \in I} (y + x \wedge c_i) = y + x$ . This proves (iii). Therefore  $c \in \Gamma(E)$ , and by [25, §20.1],  $\Gamma(E)$  is a complete boolean algebra.

(2) Put  $f = e^\sim$ . Using Zorn's lemma we choose a maximal pairwise orthogonal family  $(c_i)_{i \in I}$  in  $\Gamma(E) \cap E[0, f]$ . As  $c_i \leq f$  for all  $i \in I$ , we have  $c := \bigvee_{i \in I} c_i \leq f$ , and  $c \in \Gamma(E)$  by part (i) of this proof. Then  $d := c^- = \bigwedge_{i \in I} c_i^-$ , and  $e = f^- \leq c^- = d \in \Gamma(E)$ . To show that  $d$  is the smallest element in  $\Gamma(E)$  such that  $e \leq d$ , let  $e \leq k \in \Gamma(E)$ . Then  $k^\sim \leq e^\sim = f$ , so  $k^\sim \wedge d \in \Gamma(E) \cap E[0, f]$ . Then  $k^\sim \wedge d \leq d = c^- \leq c_i^- = c_i'$  for all  $i \in I$ , hence  $k^\sim \wedge d$  is orthogonal to all  $c_i, i \in I$ , and by maximality of  $(c_i)_{i \in I}$ ,  $k^\sim \wedge d = k' \wedge d = 0$ . Since  $k, d \in \Gamma(E)$ ,  $d \leq k$ , which proves (2).  $\square$

**4.8. Definition.** If  $e \in E$ , then the smallest element  $d \in \Gamma(E)$  such that  $e \leq d$  (Theorem 4.7 (2)) is called the *central cover* of  $e$ , and we shall denote it by  $\gamma e := d$ .

In the following definition, we extend the notion of a hull mapping [10, 12] to pseudo-effect algebras.

**4.9. Definition.** A mapping  $\eta: E \rightarrow \Gamma(E)$  such that (1)  $\eta 0 = 0$ , (2)  $e \in E \Rightarrow e \leq \eta e$ , and (3)  $e, f \in E \Rightarrow \eta(e \wedge \eta f) = \eta e \wedge \eta f$  is called a *hull mapping* on  $E$ .

**4.10. Theorem.** The central cover mapping  $\gamma: E \rightarrow \Gamma(E)$  is a surjective hull mapping<sup>2</sup> on  $E$ .

*Proof.* Obviously,  $\gamma 0 = 0$  and  $e \leq \gamma e$  for all  $e \in E$ . Let  $e, f \in E$  and put  $c := \gamma f$ . We have to prove that  $\gamma(e \wedge c) = \gamma e \wedge c$ . Since  $e \leq \gamma e$ , we have  $e \wedge c \leq \gamma e \wedge c$ , and hence  $\gamma(e \wedge c) \leq \gamma e \wedge c$ . Since  $c \in \Gamma(E)$ , we have  $e = (e \wedge c) \vee (e \wedge c') \leq \gamma(e \wedge c) \vee c' \in \Gamma(E)$ , whence  $\gamma e \leq \gamma(e \wedge c) \vee c'$ . It follows that  $\gamma e \wedge c \leq \gamma(e \wedge c) \wedge c \leq \gamma(e \wedge c)$ , as desired. Since  $\gamma(\gamma e) = \gamma(1 \wedge \gamma e) = \gamma 1 \wedge \gamma e = \gamma e$ , we obtain that  $\gamma E := \{\gamma e : e \in E\} = \Gamma(E)$ .  $\square$

**4.11. Lemma.** Suppose that  $(p_i)_{i \in I} \subseteq E$  is a  $\Gamma$ -orthogonal family in  $E$ . Let  $p := \bigvee_{i \in I} p_i$ , and let  $c_i := \gamma p_i$  for all  $i \in I$  with  $c = \bigvee_{i \in I} c_i$ . Then:

- (i)  $p \leq \gamma p = c \in \Gamma(E)$ .
- (ii)  $p \wedge c_i = p_i$  for all  $i \in I$ .
- (iii) If  $e \in E[0, p]$ , then  $e \wedge c_i = e \wedge p_i$  for all  $i \in I$  and  $e = \bigvee_{i \in I} (e \wedge p_i)$ .

*Proof.* Since  $(p_i)_{i \in I}$  is a  $\Gamma$ -orthogonal family,  $(c_i)_{i \in I}$  is an orthogonal family in  $\Gamma(E)$ , so  $p$  and  $c$  are well-defined. Since  $p_i \leq p$  for all  $i \in I$ , we have  $\bigvee_{i \in I} \gamma p_i = c \leq \gamma p$ . On the other hand,  $p_i \leq \gamma p_i \leq c$  implies  $\gamma p \leq c$ . This proves (i). Suppose that  $i, j \in I$ . If  $i = j$ , then  $p_i \wedge c_i = p_i \wedge \gamma p_i = p_i$ ; and if  $i \neq j$ , then  $c_i \wedge c_j = 0$ , so  $c_i \wedge p_j = 0$ . Therefore, by Theorem 3.11 (i),  $p \wedge c_i = (\bigvee_{j \in I} p_j) \wedge c_i = \bigvee_{j \in I} (p_j \wedge c_i) = p_i$ , which proves (ii). To prove (iii), suppose  $e \in E[0, p]$ . Then for each  $i \in I$ ,  $e \wedge c_i = e \wedge p \wedge c_i = e \wedge p_i$  by (ii). Thus by Corollary 4.6 (ii),  $e = e \wedge c = \bigvee_{i \in I} (e \wedge c_i) = \bigvee_{i \in I} (e \wedge p_i)$ .  $\square$

The following theorem extends Theorem 3.7 in the setting of COPEAs. Since the proof is analogous to [10, Theorem 6.14], we omit it.

<sup>2</sup>In [10], a surjective hull mapping from an effect algebra  $E$  onto  $\Gamma(E)$  (which is unique if it exists) is called a *discrete hull mapping*.

**4.12. Theorem.** Let  $(p_i)_{i \in I} \subseteq E$  be a  $\Gamma$ -orthogonal family in  $E$ , let  $p := \sum_{i \in I} p_i = \bigvee_{i \in I} p_i$ , and let  $X := \prod_{i \in I} E[0, p_i]$ . Define the mapping  $\Phi : X \rightarrow E[0, p]$  by

$$\Phi((e_i)_{i \in I}) := \sum_{i \in I} e_i = \bigvee_{i \in I} e_i \text{ for every } (e_i)_{i \in I} \in X.$$

Then  $\Phi$  is a PEA-isomorphism of  $X$  onto  $E[0, p]$  and

$$\Phi^{-1}(e) := (e \wedge \gamma p_i)_{i \in I} \text{ for all } e \in E[0, p].$$

## 5. TYPE-DETERMINING SETS

The assumption that  $E$  is a COPEA remains in force. As usual, a closure operator on the set of all subsets  $Q$  of  $E$  is a mapping  $Q \mapsto Q^c$  such that, for all  $Q, R \subseteq E$ , (1)  $Q \subseteq Q^c$ , (2)  $Q \subseteq R \Rightarrow Q^c \subseteq R^c$ , and (3)  $(Q^c)^c = Q^c$ . A subset  $Q$  is said to be *closed* (with respect to  $^c$ ) iff  $Q^c = Q$ . The intersection of closed subsets is necessarily closed. Generalizing the analogous notions for effect algebras in [11], we introduce the following closure operators:  $Q \mapsto [Q]$ ,  $Q \mapsto Q^\gamma$ ,  $Q \mapsto Q^\downarrow$ , and  $Q \mapsto Q''$ , where

- (i)  $[Q]$  is the set of all suprema of  $\Gamma$ -orthogonal families of elements of  $Q$ . We define  $[\emptyset] = \{0\}$ .
- (ii)  $Q^\gamma := \{q \wedge c : q \in Q, c \in \Gamma(E)\}$ .
- (iii)  $Q^\downarrow := \bigcup_{q \in Q} E[0, q]$ .
- (iv)  $Q' := \{e \in E : q \wedge e = 0 \ \forall q \in Q\}$ .
- (v)  $Q'' := (Q')'$ .

**5.1. Definition.** We say that a subset  $K \subseteq E$  is *type-determining* (TD) iff  $K = [K] = K^\gamma$ , and  $K$  is *strongly type-determining* (STD) iff  $K = [K] = K^\downarrow$ .

Clearly, the intersection of TD (respectively, STD) subsets of  $E$  is again TD (respectively, STD).

**5.2. Theorem.** Let  $Q \subseteq E$ . Then: (i)  $[Q^\gamma]$  is the smallest TD subset of  $E$  containing  $Q$ . (ii)  $[Q^\downarrow]$  is the smallest STD subset of  $E$  containing  $Q$ . (iii)  $Q'$  and  $Q''$  are STD subsets of  $E$ . (iv)  $Q' = [Q^\gamma]' = [Q^\downarrow]'$ .

*Proof.* Obviously,  $Q \subseteq [Q^\gamma]$  and if  $K$  is TD and  $Q \subseteq K$ , then  $[Q^\gamma] \subseteq K$ . Also,  $[[Q^\gamma]] \subseteq [Q^\gamma]$ , so to prove (i) it suffices to show that  $[Q^\gamma]^\gamma \subseteq [Q^\gamma]$ . Let  $e \in [Q^\gamma]^\gamma$ , then there exist  $d \in \Gamma(E)$  and  $p \in [Q^\gamma]$  with  $e = p \wedge d$ . As  $p \in [Q^\gamma]$ , there is a  $\Gamma$ -orthogonal family  $(p_i)_{i \in I} \subseteq Q^\gamma$  with  $p = \bigvee_{i \in I} p_i$ , and for each  $i \in I$ , we can write  $p_i = q_i \wedge d_i$  with  $q_i \in Q$  and  $d_i \in \Gamma(E)$ . Since  $e \leq p$ , by Lemma 4.11 (iii),  $e \wedge p_i$  exists for all  $i \in I$ ; moreover,  $e \wedge p_i = p \wedge d \wedge p_i = p_i \wedge d = q_i \wedge d_i \wedge d$ . As  $d_i \wedge d \in \Gamma(E)$ , it follows the  $e \wedge p_i \in Q^\gamma$  for all  $i \in I$ , and the family  $(e \wedge p_i)_{i \in I}$  is  $\gamma$ -orthogonal. Consequently, by Lemma 4.11 (iii),  $e = \bigvee_{i \in I} (e \wedge p_i) \in [Q^\gamma]$ . This proves (i). The proof of (ii) is quite similar to the proof of (i), and we omit it. To prove (iii), let  $e \in Q'$  and  $f \leq e$ . Then  $e \wedge q = 0$  for all  $q \in Q$ , whence  $f \wedge q = 0$  for all  $q \in Q$ , hence  $f \in Q'$ , so that  $Q' = Q'^\downarrow$ . Let  $(p_i)_{i \in I} \subseteq Q'$  be  $\Gamma$ -orthogonal family, and  $p = \bigvee_{i \in I} p_i$ . Then  $q \wedge p_i = 0$  for all  $q \in Q$  and all  $i \in I$ , and since  $q \wedge p \leq p$ , by Lemma 4.11 (iii),  $p \wedge q = \bigvee_{i \in I} p \wedge q \wedge p_i = 0$ , hence  $p \in Q'$ . It follows that  $Q' = [Q']$ , and  $Q'$  is STD. As  $Q'' = (Q')'$ , it follows that  $Q''$  is STD. To prove (iv), observe that  $Q \subseteq [Q^\gamma] \subseteq [Q^\downarrow]$  implies  $[Q^\downarrow]' \subseteq [Q^\gamma]' \subseteq Q'$ . Let  $e \in Q'$ , and  $(p_i)_{i \in I}$  be a  $\Gamma$ -orthogonal family of elements in  $Q^\downarrow$  with  $p = \bigvee_{i \in I} p_i$ . Then each

$p_i \leq q_i$  for some  $q_i \in Q$ , and  $e \wedge p_i \leq e \wedge q_i = 0$  for all  $i \in I$ . By Lemma 4.11(iii),  $e \wedge p = \bigvee_{i \in I} e \wedge p \wedge p_i = 0$ , which shows that  $e \in [Q^\perp]'$ , proving (iv).  $\square$

**5.3. Theorem.** *Let  $K \subseteq E$  be a TD set. Then: (i)  $K \cap \gamma K = K \cap \Gamma(E) \subseteq \gamma K \subseteq \Gamma(E)$ . (ii) There exists  $c \in \Gamma(E)$  such that  $\gamma K = \Gamma(E)[0, c]$ . (iii) There exists  $d \in \Gamma(E)$  such that  $K \cap \gamma K = \Gamma(E)[0, d]$ .*

*Proof.* We omit the proof since it is analogous to the proof of [11, Theorem 4.5].  $\square$

Obviously, for every  $c \in \Gamma(E)$ , the central interval  $\Gamma(E)[0, c] = \Gamma(E) \cap E[0, c]$  is a TD subset of  $E$ .

**5.4. Corollary.** *If  $K$  is a TD subset of  $E$ , then so are  $\gamma K$  and  $K \cap \gamma K$ .*

**5.5. Definition.** Let  $K$  be a TD subset of  $E$ . The (unique) element  $c \in \gamma K$  such that  $\gamma K = \Gamma(E)[0, c]$  (Theorem 5.3 (ii)) is denoted by  $c_K$  and is called the *type-cover* of  $K$ . The type cover  $c_{K \cap \gamma K}$  of the TD set  $K \cap \gamma K$  is called the *restricted type-cover* of  $K$ .

The following definition is analogous to [11, Definition 5.1]. The terminology is borrowed from [26, pp. 28–29].

**5.6. Definition.** Let  $K$  be a TD subset of the COPEA  $E$  and let  $c \in \Gamma(E)$ . Then:

- (i)  $c$  is type- $K$  iff  $c \in K$ .
- (ii)  $c$  is locally type- $K$  iff  $c \in \gamma K$ .
- (iii)  $c$  is purely non- $K$  iff no nonzero subelement of  $c$  belongs to  $K$ .
- (iv)  $c$  is properly non- $K$  iff no nonzero central subelement of  $c$  belongs to  $K$ .

If  $c \in \Gamma(E)$  and  $c$  is type- $K$  (respectively, locally type- $K$ , etc.), we shall also say that the direct summand  $E[0, c]$  of  $E$  is type- $K$  (respectively, locally type- $K$ , etc.).

The proof of the next theorem is omitted since it is the same as the proof of [11, Theorem 5.2].

**5.7. Theorem.** *Let  $K$  be a TD subset of  $E$  and let  $c \in \Gamma(E)$ . Then:*

- (i)  $c$  is type- $K \Leftrightarrow \Gamma(E)[0, c] \subseteq K \cap \gamma K \Leftrightarrow c \leq c_{K \cap \gamma K}$ .
- (ii) If  $K$  is STD, then  $c$  is type- $K \Leftrightarrow E[0, c] \subseteq K$ .
- (iii)  $c$  is locally type- $K \Leftrightarrow \Gamma(E)[0, c] \subseteq \gamma K \Leftrightarrow c \leq c_K$ .
- (iv)  $c$  is purely non- $K \Leftrightarrow K \cap E[0, c] = \{0\} \Leftrightarrow c \leq (c_K)'$ .
- (v)  $c$  is properly non- $K \Leftrightarrow K \cap \Gamma(E)[0, c] = \{0\} \Leftrightarrow c \leq (c_{K \cap \gamma K})'$ .
- (vi)  $c$  is both locally type- $K$  and properly non- $K \Leftrightarrow c \leq c_K \wedge (c_{K \cap \gamma K})'$ .

**5.8. Corollary.** *If  $K$  is a TD subset of  $E$  and  $c \in \Gamma(E)$ , the following conditions are equivalent: (i)  $c$  is locally type- $K$ . (ii) Every nonzero direct summand of  $E[0, c]$  contains a nonzero element of  $K$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Assume (i). Then by Theorem 5.7 (iii),  $\Gamma(E)[0, c] \subseteq \gamma K$ , hence  $c = \gamma k$  for some  $k \in K$ . Let  $0 \neq d \in \Gamma(E)[0, c]$ . Then  $k \wedge d \in K \cap E[0, d]$  with  $\gamma(k \wedge d) = \gamma k \wedge d = c \wedge d = d \neq 0$ , whence  $k \wedge d \neq 0$ .

(ii)  $\Rightarrow$  (i). Assume (ii). Then,  $c \wedge c'_K \leq c$  and if  $c \wedge c'_K \neq 0$ , there exists  $0 \neq k \in K$  with  $k \in E[0, c \wedge c'_K]$ , hence  $\gamma k \leq c'_K$ , contradicting Theorem 5.3 (ii). Therefore  $c \wedge c'_K = 0$ , whence  $c \leq c_K$ .  $\square$

**5.9. Lemma.** *If  $K$  is a TD subset of  $E$ , then  $c_{K' \cap \gamma(K')} = (c_K)'$ .*

*Proof.* We have to prove that  $K' \cap \gamma(K') = \Gamma(E)[0, (c_K)']$ . As  $K' \cap \gamma(K') = K' \cap \Gamma(E)$ , it suffices to prove that, for  $c \in \Gamma(E)$ ,  $c \in K' \Leftrightarrow c \leq (c_K)'$ , the latter inequality being equivalent to  $c \wedge c_K = 0$ . Let  $c \in \Gamma(E)$ . Suppose  $c \in K'$  and let  $k^* \in K$  be such that  $c_K = \gamma k^*$ , then  $c \wedge k^* = 0$ , whence  $c \wedge c_K = \gamma(c \wedge k^*) = 0$ . Conversely, suppose  $c \wedge c_K = 0$  and let  $k \in K$ . Then, as  $\gamma k \leq c_K$ , it follows that  $\gamma(c \wedge k) = c \wedge \gamma k = 0$ , whence  $c \wedge k = 0$ , so  $c \in K'$ .  $\square$

**5.10. Theorem.** *Let  $K$  be a TD subset of  $E$ . Then there exist unique pairwise orthogonal  $c_1, c_2, c_3 \in \Gamma(E)$  such that  $c_1 + c_2 + c_3 = 1$ ;*

$$E = E[0, c_1] \times E[0, c_2] \times E[0, c_3];$$

*$c_1$  is type- $K$ ;  $c_2$  is locally type- $K$ , but properly non- $K$ ; and  $c_3$  is purely non- $K$ . Moreover,  $c_1 = c_{K \cap \gamma K}$ ,  $c_2 = c_K \wedge (c_{K \cap \gamma K})'$ ,  $c_3 = (c_K)'$ ,*

$$K \cap \gamma K = \Gamma(E)[0, c_1], K \subseteq E[0, c_1 + c_2], \Gamma(E)[0, c_2 + c_3] \cap K = \{0\}.$$

*Proof.* Put  $c_1 := c_{K \cap \gamma K}$ ,  $c_2 := c_K \wedge (c_{K \cap \gamma K})'$ , and  $c_3 := (c_K)'$ . As  $c_{K \cap \gamma K} \leq c_K$ , we have  $c_1 + c_2 + c_3 = 1$ ,  $c_1 + c_2 = c_K$ , and  $c_2 + c_3 = (c_{K \cap \gamma K})'$ . Thus, by part (i) of Theorem 5.7 (i),  $c_1$  is of type- $K$ ; by part (vi) of Theorem 5.7,  $c_2$  is locally type- $K$  and properly non- $K$ , and by part (iv) of Theorem 5.7,  $c_3$  is purely non- $K$ . To prove uniqueness, suppose that  $c_1, c_2$  and  $c_3$  satisfy the conditions in the first part of the theorem. Then  $c_1 + c_2$  is locally type- $K$ , hence  $c_1 + c_2 \leq c_K$ , and  $c_3$  is purely non- $K$ , hence  $c_3 \leq (c_K)'$  by Theorem 5.7 (iii) and (iv). Since  $c_1 + c_2 + c_3 = 1 = c_K + (c_K)'$ , we have  $c_1 + c_2 = c_K$ , and  $c_3 = (c_K)'$ . Moreover,  $c_1$  is type- $K$ , hence  $c_1 \leq c_{K \cap \gamma K}$ ,  $c_2$  is locally type- $K$  but properly non- $K$ , hence  $c_2 \leq c_K \wedge (c_{K \cap \gamma K})'$ . Since  $c_1 + c_2 = c_K = c_{K \cap \gamma K} + c_K \wedge (c_{K \cap \gamma K})'$ , we obtain  $c_1 = c_{K \cap \gamma K}$ ,  $c_2 = c_K \wedge (c_{K \cap \gamma K})'$ .  $\square$

## 6. EXAMPLES OF TD SETS AND DIRECT DECOMPOSITIONS

Recall that an *atom* in a pseudo-effect algebra  $E$  is a nonzero element  $a \in E$  such that if  $x \leq a$  then either  $x = 0$  or  $x = a$ . A pseudo-effect algebra  $E$  is *atomic* iff for every  $e \in E$  there is an atom  $a \leq e$ . Let  $A$  (which may be empty) denote the set of all atoms of  $E$ .

**6.1. Lemma.** *If  $a \in A$  is an atom in  $E$ , then  $\gamma a$  is an atom in  $\Gamma(E)$ . Consequently, if  $E$  is atomic, then  $\Gamma(E)$  is atomic.*

*Proof.* Let  $a \in A$ , and  $c \in \Gamma(E)$ ,  $c \leq \gamma a$ . Then  $c = \gamma(c \wedge a)$ , so that  $c = 0$  if  $c \wedge a = 0$ , or  $c = \gamma a$  if  $c \wedge a = a$ . If  $E$  is atomic, then for every  $c \in \Gamma(E) \subseteq E$  there is  $a \in A$  with  $a \leq c$ , which yields  $\gamma a \leq c$ .  $\square$

We say that an element  $p \in E$ , or equivalently, that  $E[0, p]$  is *atom free* iff  $A \cap E[0, p] = \emptyset$ .

**6.2. Lemma.**  *$[A]$  is the STD subset of  $E$  generated by  $A$ .*

*Proof.* If  $A = \emptyset$ , then  $A^\perp = \emptyset$ , otherwise  $A^\perp = A \cup \{0\}$ . In both cases,  $[A^\perp] = [A]$ , and the result follows from Theorem 5.2 (ii).  $\square$

An element of the STD set  $[A]$  is called a *polyatom*. The following theorem for COPEAs is analogous to [11, Theorem 7.4] for COEAs, and it enables us to decompose  $E$  into atomic and atom free parts.

**6.3. Theorem.** (i) The set  $A' = [A]'$  is STD and consists of all atom free elements of  $E$ . (ii) The set  $A'' = [A]''$  is STD and its nonzero part consists of elements  $p \in E$  such that  $E[0, p]$  is atomic. (iii)  $c_{A' \cap \gamma(A')} = c'_{[A]}$  is atom free. (iv)  $A \subseteq [A] \subseteq E[0, c_{[A]}]$ . (v) If  $p \in E$ , then  $p$  is atom free iff  $[A] \cap E[0, p] = \{0\}$ . (vi)  $[A \cap \Gamma(E)] = [A] \cap \Gamma(E)$ .

*Proof.* By Theorem 5.2 (iii),  $A'$  and  $A''$  are STD subsets of  $E$ . Since  $p \in A'$  iff  $p \wedge a = 0$  for all atoms  $a \in A$ ,  $A'$  is the set of all atom free elements. Let  $p \in A''$ , then  $q \wedge a = 0$  for all  $a \in A$  implies  $q \wedge p = 0$ , hence if  $p \wedge a = 0$  for all  $a \in A$ , then  $p = 0$ . Therefore if  $0 \neq p \in A''$  then there is an atom  $a \in A$  with  $a \leq p$ . This proves (i) and (ii). Part (iii) follows from (i) and Lemma 5.9. (iv) If  $a$  is an atom, then  $a = (a \wedge c_{[A]}) + (a \wedge c'_{[A]})$ , where  $a \wedge c'_{[A]} = 0$  by part (iii). It follows that  $a \leq c_{[A]}$ . Therefore,  $A \subseteq E[0, c_{[A]}]$ , and since  $E[0, c_{[A]}]$  is STD,  $[A] \subseteq E[0, c_{[A]}]$ . (v) Every atom is a nonzero polyatom, and a polyatom is nonzero iff it dominates an atom, hence  $A \cap E[0, p] = \emptyset \Leftrightarrow [A] \cap E[0, p] = \{0\}$ . (vi) Since  $[A]$  is a TD subset of  $E$ , so is  $[A] \cap \gamma[A] = [A] \cap \Gamma(E)$ . Thus, as  $A \cap \Gamma(E) \subseteq [A] \cap \Gamma(E)$ , we have  $[A \cap \Gamma(E)] \subseteq [A] \cap \Gamma(E)$ . Let  $h \in [A] \cap \Gamma(E)$ . Since  $h \in [A]$ , there is a  $\Gamma$ -orthogonal sequence  $(a_i)_{i \in I}$  of atoms with  $h = \sum_{i \in I} a_i = \bigvee_{i \in I} a_i$ . Then  $\gamma a_i, i \in I$ , are pairwise orthogonal elements in  $\Gamma(E)$ , and since  $h \in \Gamma(E)$ ,  $h = \gamma h = \bigvee_{i \in I} \gamma a_i = \sum_{i \in I} \gamma a_i$ . It follows that  $\sum_{i \in I} a_i = \sum_{i \in I} \gamma a_i$ , and from  $a_i \leq \gamma a_i$  for all  $i \in I$ , we deduce that  $a_i = \gamma a_i \in \Gamma(E)$ , and therefore  $h \in [A \cap \Gamma(E)]$ .  $\square$

The notions of boolean and subcentral elements and monads were introduced in [10], and they also make sense in the setting of pseudo-effect algebras.

**6.4. Definition.** An element  $b \in E$  is *boolean* iff  $E[0, b]$  is a boolean algebra, i.e.,  $E[0, b] = \Gamma(E[0, b])$ .

By Lemma 3.13, for every  $p \in E$  and  $c \in \Gamma(E)$ , the element  $p \wedge c$  is central in  $E[0, p]$ . The next definition concerns those elements for which the converse also holds:

**6.5. Definition.** An element  $p \in E$  is *subcentral* iff for every  $d \in \Gamma(E[0, p])$ ,  $d = p \wedge c$  for some  $c \in \Gamma(E)$ .

Clearly, every central element is subcentral (Theorem 3.8), and every atom is subcentral.

**6.6. Definition.** An element  $h \in E$  is a *monad* iff for every  $e \in E[0, h]$ ,  $e = h \wedge \gamma e$ .

Notice that every atom is a monad. Similarly as in [11, Theorem 3.9], we obtain the following characterization of monads.

**6.7. Theorem.** Let  $h \in E$ . Then the following are equivalent: (i)  $h$  is a monad. (ii)  $h$  is both subcentral and boolean. (iii) For all  $e \in E[0, h]$ ,  $\gamma e = \gamma h \Rightarrow e = h$ . (iv) For all  $e \in E[0, h]$ ,  $e \sim^h, e^{-h} \leq (\gamma e)'$ . (v) For all  $e, f \in E[0, h]$ ,  $e +_h f$  exists  $\Leftrightarrow \gamma e \wedge \gamma f = 0$ .

*Proof.* (i)  $\Rightarrow$  (ii). Let  $h$  be a monad. Since  $\Gamma(E[0, h]) \subseteq E[0, h]$ , if  $d \in \Gamma(E[0, h])$ , then  $d = h \wedge \gamma d$ , which shows that  $h$  is subcentral. Since  $e \in E[0, h]$  implies  $e = h \wedge \gamma e$ , and  $\gamma e \in \Gamma(E)$ , by Lemma 3.13,  $e$  is central in  $E[0, h]$ , hence  $E[0, h] = \Gamma(E[0, h])$ , so  $h$  is boolean.

(ii) $\Rightarrow$ (i). As  $h$  is subcentral, every  $d \in \Gamma(E[0, h])$  is of the form  $d = h \wedge c$  for some  $c \in \Gamma(E)$ . Then  $d \leq c$  implies  $\gamma d \leq c$ , and  $d = d \wedge \gamma d = h \wedge c \wedge \gamma d = h \wedge \gamma d$ . Since  $h$  is also boolean,  $\Gamma(E[0, h]) = E[0, h]$ , whence  $d = h \wedge \gamma d$  holds for all  $d \in E[0, h]$ .

(i) $\Rightarrow$ (iii). Assume  $\gamma e = \gamma h$ ,  $e \leq h$ . Then  $e = h \wedge \gamma e = h \wedge \gamma h = h$ .

(iii) $\Rightarrow$ (iv). Assume (iii), let  $e \in E[0, h]$  and put  $f := e + (h \wedge (\gamma e)')$ . As  $e \leq \gamma e$ ,  $h \wedge (\gamma e)' \leq (\gamma e)'$ , and  $\gamma e \in \Gamma(E)$ , it follows that  $f = e \vee (h \wedge (\gamma e)') \in E[0, h]$ . Since  $\gamma e \leq \gamma h$ , we have

$$\gamma f = \gamma e \vee \gamma(h \wedge (\gamma e)') = \gamma e \vee (\gamma h \wedge (\gamma e)') = (\gamma h \wedge \gamma e) \vee (\gamma h \wedge (\gamma e)') = \gamma h,$$

whence by (iii),  $e + (h \wedge (\gamma e)') = f = h = e + e/h$  and it follows that  $h \wedge (\gamma e)' = e/h = e^{\sim h} \leq (\gamma e)'$ . We can also write  $f = (h \wedge (\gamma e)') + e = h = h \setminus e + e$ , which yields  $h \wedge (\gamma e)' = h \setminus e = e^{\sim h} \leq (\gamma e)'$ .

(iv) $\Rightarrow$ (v). Let  $e, f \in E[0, h]$ , and assume that  $e +_h f$  exists. Then  $f \leq e^{\sim h} \leq (\gamma e)'$ , the last inequality following from (iv). Now  $f \leq (\gamma e)'$  implies  $\gamma f \leq (\gamma e)'$  which entails (v).

(v) $\Rightarrow$ (i). Let  $e \in E[0, h]$ , then  $h = e + e^{\sim h} = e + (e/h)$ , and by (v),  $\gamma(e/h) \leq (\gamma e)'$ . We also have  $h = h \wedge \gamma e + h \wedge (\gamma e)'$ , and from  $e \leq h \wedge \gamma e$  and  $e/h \leq h \wedge (\gamma e)'$  we deduce that  $e = h \wedge \gamma e$ , whence  $h$  is a monad.  $\square$

Let  $S$  denote the set of all subcentral elements of  $E$ ,  $B$  the set of all boolean elements of  $E$  and  $H$  the set of all monads in  $E$ . As in [10], it can be shown that  $S$  is a TD set with  $[A] \subseteq S$ ,  $B$  is an STD set with  $[A] \subseteq B$ , and  $H = S \cap B$  is an STD set with  $[A] \subseteq H$ .

The following definition is an analogue of [11, Definition 4.2].

**6.8. Definition.** A nonempty class  $\mathcal{K}$  of PEAs is called a *type-class* iff the following conditions are satisfied: (1)  $\mathcal{K}$  is closed under the passage to direct summands, i.e., if  $H \in \mathcal{K}$  and  $h \in \Gamma(H)$ , then  $H[0, h] \in \mathcal{K}$ . (2)  $\mathcal{K}$  is closed under the formation of arbitrary direct products. (3) If  $E_1$  and  $E_2$  are isomorphic PEAs and  $E_1 \in \mathcal{K}$ , then  $E_2 \in \mathcal{K}$ . If, in addition to (2) and (3),  $\mathcal{K}$  satisfies (1')  $H \in \mathcal{K}$ ,  $h \in H \Rightarrow H[0, h] \in \mathcal{K}$ , then  $\mathcal{K}$  is called a *strong type-class*.

We omit the proof of the next theorem as it is analogous to the proof of [11, Theorem 4.4].

**6.9. Theorem.** Let  $\mathcal{K}$  be a type-class of COPEAs and define  $K := \{k \in E : E[0, k] \in \mathcal{K}\}$ . Then  $K$  is a TD subset of  $E$ . If  $\mathcal{K}$  is a strong type-class, the  $K$  is STD.

**6.10. Examples.** The class of effect algebras (EAs) and the following subclasses of effect algebras are strong type-classes: all boolean EAs, all OMLs, all complete OMLs, all orthoalgebras, all lattice EAs, and all atomic EAs. Similarly, all lattice-ordered PEAs and all atomic PEAs are strong type-classes.

According to [5], the PEA  $E$  is (i) *monotone  $\sigma$ -complete* iff any ascending sequence  $x_1 \leq x_2 \leq \dots$  in  $E$  has a supremum  $\bigvee_{i=1}^{\infty} x_i$  in  $E$ ; (ii)  $E$  is  *$\sigma$ -complete* iff it is a  $\sigma$ -complete lattice; (iii)  $E$  satisfies the *countable Riesz interpolation property* ( $\sigma$ -RIP) iff, for countable sequences  $\{x_1, x_2, \dots\}$  and  $\{y_1, y_2, \dots\}$  of elements of  $E$  such that  $x_i \leq y_j$  for all  $i, j$ , there exists an element  $z \in E$  such that  $x_i \leq z \leq y_j$  for all  $i, j$ ; and (iv)  $E$  is *archimedean* iff the only  $x \in E$  such that  $nx := x + \dots + x$  is defined in  $E$  for any integer  $n \geq 1$  is  $x = 0$ .

One can easily deduce that the monotone  $\sigma$ -complete PEAs, the  $\sigma$ -complete PEAs, the PEAs with the countable Riesz interpolation property, and archimedean PEAs are all strong type-classes.

In [6], the following properties of PEAs were introduced.

**6.11. Definition.** Let  $(E; +, 0, 1)$  be a pseudo-effect algebra. Then:

- (i)  $E$  fulfills the *Riesz Interpolation Property* (RIP) iff, for any  $a_1, a_2, b_1, b_2 \in E$  such that  $a_1, a_2 \leq b_1, b_2$  there is  $c \in E$  such that  $a_1, a_2 \leq c \leq b_1, b_2$ .
- (ii)  $E$  fulfills the *Weak Riesz Decomposition Property* ( $\text{RDP}_0$ ) iff, for any  $a, b_1, b_2 \in E$  such that  $a \leq b_1 + b_2$ , there are  $d_1, d_2 \in E$  such that  $d_1 \leq b_1, d_2 \leq b_2$  and  $a = d_1 + d_2$ .
- (iii)  $E$  fulfills the *Riesz Decomposition Property* (RDP) iff, for any  $a_1, a_2, b_1, b_2 \in E$  such that  $a_1 + a_2 = b_1 + b_2$  there are  $d_1, d_2, d_3, d_4 \in E$  such that  $d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1$ , and  $d_2 + d_4 = b_2$ .
- (iv)  $E$  fulfills the *Commutational Riesz Decomposition Property* ( $\text{RDP}_1$ ) iff, for any  $a_1, a_2, b_1, b_2 \in E$  such that  $a_1 + a_2 = b_1 + b_2$  there are  $d_1, d_2, d_3, d_4 \in E$  such that (1)  $d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1, d_2 + d_4 = b_2$  and (2)  $x \leq d_2, y \leq d_3$  imply  $x + y = y + x$ .
- (v)  $E$  fulfills the *Strong Riesz Decomposition Property* ( $\text{RDP}_2$ ) iff, for any  $a_1, a_2, b_1, b_2 \in E$  such that  $a_1 + a_2 = b_1 + b_2$  there are  $d_1, d_2, d_3, d_4 \in E$  such that (1)  $d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1, d_2 + d_4 = b_2$  and (2)  $d_2 \wedge d_3 = 0$ .

**6.12. Proposition.** [6, Proposition 3.3] *Let  $(E; +, 0, 1)$  be a pseudo-effect algebra.*

(i) *We have the implications*

$$(\text{RDP}_2) \Rightarrow (\text{RDP}_1) \Rightarrow (\text{RDP}) \Rightarrow (\text{RDP}_0) \Rightarrow (\text{RIP}),$$

*The converse of any of these implications fails.*

(ii)  *$E$  fulfils  $(\text{RDP}_2)$  iff  $E$  is lattice ordered and fulfils  $(\text{RDP}_0)$ .*

(iii) *Let  $E$  be commutative (i.e., an effect algebra) Then we have the implications*

$$(\text{RDP}_2) \Rightarrow (\text{RDP}_1) \Leftrightarrow (\text{RDP}) \Leftrightarrow (\text{RDP}_0) \Rightarrow (\text{RIP}).$$

*Any implication not shown here does not hold.*

Since for any  $k \in E$ , if  $a + b$  exists in  $E[0, k]$  then  $a + b$  exists in  $E$ , and the operations in direct products are defined pointwise, it is easy to deduce that PEAs with any of the properties from Definition 6.11 are strong type-classes.

In [27], the following class of PEAs was introduced: An effect algebra  $E$  is *weak-commutative* if, for any  $a, b \in E$ ,  $a + b$  exists iff  $b + a$  exists. It is easy to see that  $E$  is weak-commutative iff for all  $a \in E$ ,  $a^- = a^\sim$ . Indeed, if  $E$  is weak-commutative, from  $a^- + a = 1 = a + a^\sim$  we obtain  $a + a^-$  and  $a^\sim + a$  exist, so that  $a^- \leq a^\sim$  and  $a^\sim \leq a^-$ . On the other hand, if  $a^- = a^\sim$ , then  $a + b$  exists iff  $b \leq a^\sim = a^-$  iff  $b + a$  exists. A weak-commutative PEA becomes an effect algebra iff  $a + b = b + a$  whenever one side of the equality exists. It was shown in [27] that effect algebras are a proper subclass of weak-commutative pseudo-effect algebras.

**6.13. Theorem.** *The class of weak-commutative PEAs is a type-class, which is not a strong type-class.*

*Proof.* Let  $c \in \Gamma(E)$ ,  $a, b \in E[0, c]$ . Then  $a + b$  exists in  $E[0, c]$  iff  $a + b$  exists in  $E$ , so  $b + a$  exists in  $E$ , whence  $b + a$  exists in  $E[0, c]$ . Verification of the remaining properties of a type-class is straightforward. Suppose that the class in question is a

strong type-class. Then for every  $d \in E$ ,  $E[0, d]$  would be weak-commutative; hence if  $a, b \leq d$  and  $a + b \leq d$ , then  $b + a \leq d$ . Putting  $d = a + b$  yields  $b + a \leq a + b$ , and putting  $d = b + a$  yields  $a + b \leq b + a$ .  $\square$

In what follows we assume that  $K$  and  $F$  are TD subsets of the COPEA  $E$  and that  $K \subseteq F$ . As in Theorem 5.10, we decompose  $E$  as

$$E = E[0, c_1] \times E[0, c_2] \times E[0, c_3] \text{ and also as } E = E[0, d_1] \times E[0, d_2] \times E[0, d_3]$$

where  $c_1 = c_{K \cap \gamma K}$  and  $d_1 = c_{F \cap \gamma F}$  are of types  $K$  and  $F$ , respectively;  $c_2 = c_K \wedge c'_{K \cap \gamma K}$  and  $d_2 = c_F \wedge c'_{F \cap \gamma F}$  are locally types  $K$  and  $F$ , but properly non- $K$  and properly non- $F$ , respectively; and  $c_3 = c'_K$  and  $d_3 = c'_F$  are purely non- $K$  and purely non- $F$ , respectively.

As  $K \subseteq F$ , it is clear that, type- $K$  implies type- $F$ ; locally type- $K$  implies locally type- $F$ ; properly non- $F$  implies properly non- $K$ ; and properly non- $F$  implies properly non- $K$ .

The following theorem is an analogue of [11, Theorem 6.6] proved for effect algebras; since its proof in pseudo-effect algebra setting follows the same ideas, we omit it.

**6.14. Theorem.** *There exists a direct sum decomposition*

$$E = E[0, c_{11}] \times E[0, c_{21}] \times E[0, c_{22}] \times E[0, c_{31}] \times E[0, c_{32}] \times E[0, c_{33}]$$

where  $c_{11}$  is type- $K$  (hence type- $F$ );  $c_{21}$  is type- $F$ , locally type- $K$ , but properly non- $K$ ;  $c_{22}$  is locally type- $K$  (hence, locally type- $F$ ), but properly non- $F$  (hence, properly non- $K$ );  $c_{31}$  is type- $F$  and purely non- $K$ ;  $c_{32}$  is locally type- $F$  but properly non- $F$ , and purely non- $K$ ; and  $c_{33}$  is purely non- $F$  (hence, purely non- $K$ ). Moreover, such a decomposition is unique, with  $c_{ij} = c_i \wedge d_j$  for  $i, j = 1, 2, 3$ , where  $c_{11} = c_1$ ,  $c_{33} = d_3$  and  $c_{12} = c_{13} = c_{23} = 0$ .

In analogy with the classical decomposition of von Neumann algebras into types I, II, and III, we introduce the following definition (see also [11, Definition 6.3]).

**6.15. Definition.** For the TD sets  $K$  and  $F$  with  $K \subseteq F$ , the COPEA  $E$  is *type I* iff it is locally type- $K$ ; *type II* iff it is locally type- $F$ , but purely non- $K$ ; and *type III* iff it is purely non- $F$ . It is type  $I_F$  (respectively, type  $II_F$ ) iff it is type I (respectively, type II) and also type- $F$ . It is type  $I_{\bar{F}}$  (respectively, type  $II_{\bar{F}}$ ) iff it is of type I (respectively, type II) and also properly non- $F$ .

The following theorem is the I/II/III-decomposition theorem for COPEAs.

**6.16. Theorem.**  *$E$  decomposes as  $E = E[0, c_I] \times E[0, c_{II}] \times E[0, c_{III}]$ , where  $c_I, c_{II}$  and  $c_{III}$  are central elements of types I, II, and III, respectively; such a decomposition is unique, and  $c_I = c_K$ ,  $c_{II} = c_F \wedge c'_K$ ,  $c_{III} = c'_F$ .*

*Moreover, there are further decompositions  $E[0, c_I] = E[0, c_{IF}] \times E[0, c_{I\bar{F}}]$  and  $E[0, c_{II}] = E[0, c_{IIF}] \times E[0, c_{II\bar{F}}]$ , where  $c_{IF}, c_{I\bar{F}}, c_{IIF}, c_{II\bar{F}}$  are central elements of types  $I_F, I_{\bar{F}}, II_F, II_{\bar{F}}$ , respectively; these decompositions are also unique.*

These decompositions are obtained if in Theorem 6.14 we put  $c_I := c_{11} + c_{21} + c_{22}$ ;  $c_{II} := c_{31} + c_{32}$  and  $c_{III} = c_{33}$ ;  $c_{IF} := c_{11} + c_{21}$ ,  $c_{I\bar{F}} := c_{22}$ ,  $c_{IIF} := c_{31}$ ,  $c_{II\bar{F}} := c_{32}$ . Notice that, beyond the traditional I/II/III decomposition, the type  $I_F$  summand decomposes as  $E[0, c_{IF}] = E[0, c_{11}] \times E[0, c_{21}]$ , where  $c_{11}$  is type- $K$  (hence type- $F$ ) and  $c_{21}$  is type- $F$  and locally type- $K$ , but properly non- $K$ .



**6.17. Example.** Taking  $K := [A]$ , the set of all polyatoms, and  $F := H$ , the set of all monads of  $E$ , in Theorem 6.16, we have  $[A] \subseteq H$ , and  $E$  decomposes as  $E = E[0, r_1] \times E[0, r_2] \times E[0, r_3]$  where every nonzero direct summand of  $E[0, r_1]$  contains an atom;  $E[0, r_2]$  is atom free, but every nonzero direct summand of  $E[0, r_2]$  contains a nonzero monad; and  $E[0, r_3]$  contains no nonzero monad. This decomposition is unique. Indeed,  $r_1 = c_{[A]}$  is locally type- $[A]$ ;  $r_2 = c_H \wedge c'_{[A]}$  is locally type- $H$  and purely non- $[A]$ , and  $r_3 = c'_H$  is purely non- $H$  (see Theorem 6.16 and Corollary 5.8).

**6.18. Example.** Take  $K =: EA$ , the subset of all elements  $e \in E$  such that  $E[0, e]$  is commutative PEA (i.e., an effect algebra), and  $F =: W$ , the set of all elements  $d \in E$  such that  $E[0, d]$  is weak-commutative. Then  $EA \subseteq W$ , and we obtain the decomposition  $E = E[0, v_1] \times E[0, v_2] \times E[0, v_3]$ . The summand  $E[0, v_1]$  is locally commutative in the sense that  $v_1 = \gamma e = c_{EA}$ ; the summand  $E[0, v_2]$  is locally weak-commutative, but purely non-commutative, that is,  $v_2 = c_W \wedge c'_{EA}$ ; and  $E[0, v_3]$  is purely non-weak-commutative, that is,  $v_3 = c'_W$ . We recall that then every direct sub-summand of  $E[0, v_1]$  contains an element  $e \in EA$ ; every direct sub-summand of  $E[0, v_2]$  contains an element  $d \in W$ , but  $E[0, v_2] \cap EA = \{0\}$ ; and  $E[0, v_3]$  contains no element of  $W$ .

The summands  $E[0, v_1]$  and  $E[0, v_2]$  decompose further into weak-commutative and properly non-weak-commutative parts; and the weak-commutative part of  $E[0, v_1]$  admits a further decomposition into a commutative and a locally commutative, but properly non-commutative parts.

Let  $R2$  denote the STD of elements  $e \in E$  such that  $E[0, e]$  satisfies  $(RDP_2)$  and  $L$  denote the set of elements  $e \in E$  such that  $E[0, e]$  is a lattice.

**6.19. Example.** There exists a decomposition  $E = E[0, c_{11}] \times E[0, c_{21}] \times E[0, c_{22}] \times E[0, c_{31}] \times E[0, c_{32}] \times E[0, c_{33}]$  where  $E[0, c_{11}]$  satisfies  $(RDP_2)$ , hence is a lattice;  $E[0, c_{21}]$  is a lattice, every direct sub-summand contains an element from  $R2$ , but no direct sub-summand satisfies  $(RDP_2)$ ;  $E[0, c_{22}]$  contains no lattice ordered direct sub-summand (hence no sub-summand satisfying  $(RDP_2)$ ), but every direct sub-summand contains an element from  $R2$  (hence from  $L$ );  $E[0, c_{31}]$  is a lattice and contains no element from  $R2$ ;  $E[0, c_{32}]$  contains no lattice ordered direct sub-summand, and no element from  $R2$ , but every direct sub-summand contains an element from  $L$ ; and  $E[0, c_{33}]$  contains no element from  $L$  (hence no element from  $R2$ ). Moreover, such a decomposition is unique.

Indeed, such a decomposition is obtained from decompositions corresponding to STD sets  $R2$  and  $L$  as in Theorem 6.16, taking into account that  $R2 \subseteq L$  by proposition 6.12 (ii).

Notice that by [7], a pseudo-effect algebra satisfying  $(RDP_2)$  is a pseudo-MV algebra (a non-commutative analogue of an MV-algebra, see [23, 13]).

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